Updated: May 14, 2014

Review For Final

The directions for the exam are as follows:

"WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!"

1. The exam consists of 10 core problems and 4 extra-credit problems. If you wish, you can do all the 14 problems, but your score will only add up to 100 points. Partial credit will be given.

2. You are allowed to use a scientific calculator. Don't forget to bring it.

3. When you are studying for this exam, be sure to work through sections that you know least of all first.

4. Odd exercises have solutions at the back of your textbook.

Warning! Be sure to work on ALL exercises below that are marked in red. 100% of regular exam questions will consist of a subset of the red problems. Do ALL the problems on the review list to insure a perfect mastery of the topic.

Review of First Semester Calculus

- Suppose F(x) is the antiderivative of f(x). Explain why every other antiderivative of f(x) must be of the form F(x) + C, where C is any constant.
- Let f(x) be any continuous function. Argue that f(x) must have an anti-derivative. In particular, explain why the function $F(x) = \int_a^x f(t)dt$ represents an antiderivative of f(x). Why is it important to replace x by t in the integrand?
- Explain why the familiar procedure for solving definite integrals works. For instance, why $\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2}$?
- Let $f(x) = x^2$. Find
  (a) $k'(x)$, where $k(x) = \int_1^x f(t)dt$
  (b) $l'(x)$, where $l(x) = \int_1^x f(x)dt$
  (c) Does the definition $m(x) = \int_1^x f(x)dx$ make sense?
- Find a simpler expression for the following functions

  (a) $F(x) = \int_1^{e^x} \frac{\sin(\ln t)}{t} dt$

  (b) $G(x) = \int_0^{x^2} \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$

(c) $H(x)=\int_{-\tan x}^{\tan x}\frac{1}{1+t^2}\,dt$.

- Use Riemann sums to justify
  (a) $\int_a^b \big(f(x)+g(x)\big)dx=\int_a^b f(x)dx+\int_a^b g(x)dx$
  (b) $\left|\int_a^b f(x)dx\right|\le\int_a^b |f(x)|dx$

- Calculate by any means
  (a) $\lim_{n\to\infty}\left(\frac{2}{n}\left[3+\frac{2}{n}\right]^2+\frac{2}{n}\left[3+2\frac{2}{n}\right]^2+\cdots+\frac{2}{n}\left[3+n\frac{2}{n}\right]^2\right)$
  (b) $\lim_{n\to\infty}\left(-\frac{7}{n}\left[\frac{1}{2\sqrt{16-\frac{7}{n}}}\right]-\frac{7}{n}\left[\frac{1}{2\sqrt{16-2\frac{7}{n}}}\right]-\cdots-\frac{7}{n}\left[\frac{1}{2\sqrt{16-n\frac{7}{n}}}\right]\right)$

Section 7.1

- Know how to apply integration by parts to solve definite and indefinite integrals (P. 468-469, Exercises **3, 5, 11, 17**).
- Use integration by parts to establish reduction formulas. (P. 469, Exercises **51, 53**)

Section 7.2

- Know how to solve trigonometric integrals (P. 476-477, Exercises **1, 3, 7, 11, 29, 33, 41, 43**)

Section 7.3

- Be able to recognize integrals that can be solved with a trigonometric substitution. (P. 483, Exercises **5, 9, 13, 17**)

Section 7.4

- Be able to solve integrals of rational functions (P. 492-493, Exercises **7, 9, 15, 19, 21, 23**).
- Know how to make an appropriate substitution to rationalize the integrand (P. 493, Exercises **39, 47, 51**).

Section 7.8

- Be able to determine whether an integral is improper. Decide when an improper integral is convergent and when it is divergent. Whenever possible, find the value of the integral (P. 527, Exercises **5, 13, 15, 17, 21, 23**)
- Be able to use comparison test for integrals to check for convergence (P. 528, Exercises **49, 53**)

Section 11.1

- Know how to find a formula for the general term of a sequence from the pattern you are presented with (P. 700, Exercises **13, 15, 17**)
- Determine the limit of the sequence if it exists. (P. 700, Exercises **23, 25, 29, 31, 43, 45, 47**).
- Determine whether given sequence is monotonic and bounded. (P 701, Exercises **73, 75, 77**)
- Be able to use the monotone convergence theorem to find limits of monotone, bounded sequences (P. 701, Exercises **79, 81**)
- Calculate the following limits:

  (a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

  (b) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n$

  (c) $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n$

  (d) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

  (e) $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n$

  (f) $\lim_{n\to\infty}\left(1+\sin\left(\frac{\pi}{n}\right)\right)^{\frac{2}{\sin\left(\frac{\pi}{n}\right)}}$

  (g) $\lim_{n\to\infty}\left(1+\sin\left(\frac{\pi}{n}\right)\right)^n$
- **Possible Extra-Credit:** Establish the identity $\lim_{h\to 0}(1+h)^{1/h}=e$ from your knowledge of the derivative of $\ln(x)$.

Section 11.2

- Be able to identify convergent geometric series and find their sum (P. 711, Exercises **23, 25**).
- Determine whether the given series is convergent or divergent. Find the sum for convergent series (P. 711, Exercises **27, 29, 31, 39, 41**).
- **Possible Extra-Credit:** Establish criteria for convergence of geometric series. If a geometric series is convergent, show how to compute its sum. Justify your calculations (i.e. show me that you understand and haven't simply regurgitated the information back at me).
- **Possible Extra-Credit:** A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters. Assuming the ball continuous to bounce indefinitely, find the total distance it travels.

Section 11.3

- Be able to use the integral test to determine convergence or divergence of series (P. 720, Exercises **3, 9, 15, 17, 19, 23, 25**).
- Identify all values of p, for which the given series will be convergent (P. 721, Exercises **29-32**).

- Be able to use your knowledge of the sum of a series to determine the sum of a similar series (P. 721, Exercises **34, 35**).
- **Possible Extra-Credit:** State and prove the integral test.

Section 11.4

- Understand and be able to use the comparison and limit comparison tests (P. 726, Exercises **3, 5, 7, 17, 21, 23, 27, 31**).
- **Possible Extra-Credit:** State and prove the comparison test.
- **Possible Extra-Credit:** State and prove the limit comparison test.
- **Possible Extra-Credit:** Solve Exercises **40, 41** on page 727 for deeper understanding of the comparison and limit comparison tests.

Section 11.5

- Be able to use the alternating series test to determine convergence or divergence of series (P. 731, Exercises **3, 5, 7, 9, 15, 17, 19**).
- **Possible Extra-Credit:** State and prove the alternating series test.
- **Possible Extra-Credit:** Establish the alternating series estimation theorem (P. 730)

Section 11.6

- Be able to use the ratio, root, and absolute convergence tests to determine convergence or divergence of series (P. 737-738, Exercises **3, 7, 9, 19, 21, 23, 25**).
- **Possible Extra-Credit:** State and prove the absolute convergence test.
- **Possible Extra-Credit:** State and prove the ratio test.
- **Possible Extra-Credit:** State and prove the root test.

Section 11.7

- Be able to use everything you learned to test for convergence or divergence of series (P. 740-741, Exercises **1, 3, 7, 9, 19, 21, 23, 25, 27**).

Advanced Convergence/ Divergence Practice Problems

- One of the problems below will be featured on the final

(a) Find the exact sum of the series $\sum_{n=1}^{\infty} \frac{2^n}{b_n}$ where $b_n = \int_0^{\infty} y^n e^{-y} dy$.

(b) For what values $\alpha \in \mathbb{R}$, if any, does the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\alpha}}$ converge?

(c) Find the exact sum of $\sum_{n=0}^{\infty} r^n$ where $r = \dfrac{1}{\int_0^{\pi/3} \frac{\tan^3 x}{\cos^3 x} dx}$.

(d) Determine if $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

(e) Determine if $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ converges.

(f) Determine if $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^3$ converges.

(g) Determine if $\sum_{n=2}^{\infty} \frac{1}{n + n\cos^2 n}$ converges.

(h) Determine if $\sum_{n=1}^{\infty} Sin\left(\frac{1}{n}\right)$ converges.

(i) Find the limit of the sequence $a_n = \int_0^9 (x-1)^{-\frac{1}{n}} dx$ or prove that the limit does not exist.

(j) Find the limit of the sequence $a_n = \int_0^{\pi/2} e^{-nx} \cos x dx$ or prove that the limit does not exist.

(k) Find the limit $\lim_{n\to\infty} \sqrt[n]{n!}$

Section 11.8

- Know how to find the radius and interval of convergence of a given power series (P. 745-746, Exercises **3, 7, 13, 15, 19, 23, 27**)
- **Possible Extra-Credit:** P. 746, Exercises **29, 31, 33**
- Suppose that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$ by the root test.

  What can you say about the radius of convergence of $\sum_{n=0}^{\infty} (b_n)^2 x^n$?

Section 11.9

- Find the power series representation (P. 751-752, Exercises **3-9, 15-19** [odd]).

- Evaluate the indefinite integral as a power series. (P. 752, Exercises **25, 27**).
- **Possible Extra-Credit:** Prove or disprove. If *f* has a power series representation $\sum_{n=0}^{\infty} a_n x^n$ in the interval (-r, r), then *f* has derivatives of all orders in the interval (-r, r).

Section 11.10

- Be able to compute Taylor series for the following functions:
  (a) $f(x) = e^x$

  (b) $f(x) = \cos x$

  (c) $f(x) = \sin x$

  (d) $f(x) = \ln x$

  (e) $f(x) = tan^{-1}x$

- Be able to use your knowledge of basic Taylor series to quickly compute power series of similar functions.
  (a) $f(x) = xe^x$

  (b) $f(x) = \sin(6x^2)$

  (c) $f(x) = 3xtan^{-1}(x^3)$

- Find the Taylor Series. (P. 765, Exercises **13-19** [odd]).
- Evaluate the indefinite integral as an infinite series (P. 766, Exercises **47, 49**)
- Be able to use infinite series to solve limits (P. 766, Exercises **55, 56, 57**)
- Find the sum of the series (P. 766, Exercises **63-69** [odd]).
- **Possible Extra-Credit:** Prove or disprove. If *f* has a power representation $\sum_{n=0}^{\infty} a_n x^n$ in the interval (-r, r), then *f* has a Maclaurin series representation in (-r, r) with $\frac{f^{(n)}(0)}{n!} = a_n$.
- **Possible Extra-Credit:** Let $f(x) = \tan^{-1}(x^7)$. Calculate the 100th derivative of *f* at x = 0. That is, find $f^{(100)}(0)$.
- **Possible Extra-Credit:** If *f* is a function, for which the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ can be computed in some interval (-r, r), does it follow

that $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ for all x in some subinterval of (-r, r)? [Hint: Try $f(x)=\begin{cases} e^{-x^{-2}} & x\neq 0 \\ 0 & x=0 \end{cases}$. Compute the Maclaurin series of *f*. Does *f* equal to its Maclaurin series in some interval (-r, r)?]

- **Possible Extra-Credit:** Let $a > 0$. Use power series to find a function *f* satisfying $f''(x) = -af(x)$ where $f(0) = 0$ and $f'(0) = \sqrt{a}$.
- **Possible Extra-Credit:** (a) Derive the identity $e^{i\theta} = \cos\theta + i\sin\theta$
  (b) Use the identity obtained in part (a) to determine the identities for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.
- **Possible Extra-Credit:** Suppose that $f(x)$ is continuous, but nowhere differentiable on $(-\infty,\ \infty)$ (there are such functions. Trust me!). Then, by the fundamental theorem of calculus, $F(x)=\int_0^x f(t)dt$ is differentiable with derivative $f(x)$. Does F have a power series expansion about $x = 0$? How about some other point?